5.1 Areas and Distances

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1. Overview

Idea: we approximate the area under the curve by rectangles. The approximation will be better and better if we use more and more rectangles. To find the area exactly we need to take a limit.

More precisely

Say we're trying to approximate the area under the curve $y = f(x)$ on the interval $[a, b]$. Suppose we want to use n rectangles to do this. Then we divide up the integral into n even subintervals. Each of these little subintervals has length $\Delta x = \frac{b-a}{n}$. This will be the width of the rectangles. We name the left endpoint of the i^{th} interval x_{i-1} and the right endpoint x_i. Then since we start at a and add Δx each time, $x_i = a + i\Delta x$.

We have several options for how high the rectangles should go. For now we just say that x_i^* is a sample point in the i^{th} subinterval (so it's any number between x_{i-1} and x_i). So the area of the i^{th} rectangle will be:

$$A_i = (\text{height of } i^{\text{th}} \text{ rectangle}) \cdot (\text{width of } i^{\text{th}} \text{ rectangle}) = f(x_i^*) \cdot \Delta x$$

So we approximate the area under the curve by adding up the area of the rectangles:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

If we pick our sample points to be the right endpoints, $x_i^* = x_i$ and:

$$A \approx R_n = \sum_{i=1}^{n} f(x_i)\, \Delta x$$

If we pick our sample points to be the left endpoints, $x_i^* = x_{i-1}$ and:

$$A \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\, \Delta x$$

To get an exact value for the area we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

2. Examples

1.) Consider the region under the curve $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$.

(a) Estimate the area using 5 approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is this an over-estimate or under-estimate?

We divide up the interval $[0, 5]$ into five even intervals with endpoints $0, 1, 2, 3, 4, 5$. If we are using right endpoints we use $x_i = 1, 2, 3, 4, 5$. This gives us five rectangles. We add up the area of the rectangles to approximate the area under the curve.

$$A \approx A_1 + A_2 + A_3 + A_4 + A_5$$

The length Δx of each little interval is the width of the rectangle, and in this case $\Delta x = 1$. The height of each rectangle is $f(x_i)$. So the area of one rectangle is:

$$A_i = f(x_i) \cdot \Delta x = f(x_i)$$

We compute each $f(x_i)$:

$$\begin{aligned} f(1) &= 25 - (1)^2 = 24 \\ f(2) &= 25 - (2)^2 = 21 \\ f(3) &= 25 - (3)^2 = 16 \\ f(4) &= 25 - (4)^2 = 9 \\ f(5) &= 25 - (5)^2 = 0 \end{aligned}$$

So the area under the curve is approximately:

$$A \approx 24 + 21 + 16 + 9 + 0 = 70$$

If you draw the picture, you can see that all the rectangles are under the curve, so this is an under-estimate.

(b) Estimate the area using 5 approximating rectangles and *left* endpoints. Sketch the graph and the rectangles. Is this an over-estimate or under-estimate?

Since we are using left endpoints we have $x_i = 0, 1, 2, 3, 4$. Again the width of each rectangle is $\Delta x = 1$. This time the heights are:

$$\begin{aligned} f(0) &= 25 - (0)^2 = 25 \\ f(1) &= 25 - (1)^2 = 24 \\ f(2) &= 25 - (2)^2 = 21 \\ f(3) &= 25 - (3)^2 = 16 \\ f(4) &= 25 - (4)^2 = 9 \end{aligned}$$

So the area under the curve is approximately:

$$A \approx 25 + 24 + 21 + 16 + 9 = 95$$

If you draw the picture, you can see that the rectangles cover more area than just the area under the curve, so this is an over-estimate.

(c) Express the area under the curve as a limit.

Notice in (a) that what we did was this:

$$A \approx R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$

where $x_i = 0 + i\Delta x = i\Delta x$ and $\Delta x = \frac{5-0}{5} = 1$. So if we were using n intervals and right endpoints we'd have:

$$A \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $x_i = 0 + i\Delta x = i\Delta x$ and $\Delta x = \frac{5-0}{n} = \frac{5}{n}$. Substituting in:

$$\begin{aligned} A \approx R_n &= \sum_{i=1}^{n} f(i\Delta x)\,\Delta x \\ &= \sum_{i=1}^{n} f(i \cdot \tfrac{5}{n}) \cdot \tfrac{5}{n} \\ &= \sum_{i=1}^{n} f(\tfrac{5i}{n}) \cdot \tfrac{5}{n} \\ &= \sum_{i=1}^{n} (25 - (\tfrac{5i}{n})^2) \cdot \tfrac{5}{n} \end{aligned}$$

To get an exact value for the area, we just take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} (25 - (\tfrac{5i}{n})^2) \cdot \tfrac{5}{n}$$

We could have done this using left endpoints instead, because when we take the limit, the two will be the same. Notice that in (b), we did the following:

$$A \approx L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x$$

where, again $x_i = 0 + i\Delta x = i\Delta x$ and $\Delta x = \frac{5-0}{5} = 1$. So if we were using n intervals and left endpoints we would have:

$$A \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

where $\Delta x = \frac{5}{n}$. The only difference is that instead of x_i we have x_{i-1}. So we'd get:

$$A \approx L_n = \sum_{i=1}^{n} (25 - (\tfrac{5(i-1)}{n})^2) \cdot \tfrac{5}{n}$$

And the exact value of the area would be:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} (25 - (\tfrac{5(i-1)}{n})^2) \cdot \tfrac{5}{n}$$

2.) Suppose George was riding down the highway in his motorcycle, and he decided to look at his spedometer every 12 minutes and record his speed. This is the chart he comes up with:

time (min)	0	12	24	36	48	60
speed (feet/min)	30	28	25	22	24	27

Use this data to approximate the total distance he traveled.

To start, just look at the first 12 minutes. We will assume that George did not change his speed at all during those 12 minute, i.e. that he went exactly 30 feet/min the whole 12 minutes. (This is obviously not true, but we have no idea what George's speed was except at the times he recorded it, so what can we do?! We've got to make an approximation.) Ok, so that would mean that he went a distance of

$$d_1 = (\text{rate}) \times (\text{time}) = 30 \times 12 = 360 \text{ feet}$$

in the first 12 minutes. We can do the same thing for the four other time intervals:

$$\begin{aligned} d_2 &= 12 \times 28 \\ d_3 &= 12 \times 25 \\ d_4 &= 12 \times 22 \\ d_5 &= 12 \times 24 \end{aligned}$$

So the total distance is approximately:

$$d \approx d_1 + d_2 + d_3 + d_4 + d_5 = 12(30 + 28 + 25 + 22 + 24) = 1548 \text{ feet}$$

Note: we could have done this another way. Instead of using the *initial* speed to calculate the distance traveled over a particular time interval, we could have used the *final* speed. This is like the difference between using left and right endpoints. In particular if we used the final speeds we would have:

$$\begin{aligned} d_1 &= 12 \times 28 \\ d_2 &= 12 \times 25 \\ d_3 &= 12 \times 22 \\ d_4 &= 12 \times 24 \\ d_5 &= 12 \times 27 \end{aligned}$$

And we would get:

$$d \approx d_1 + d_2 + d_3 + d_4 + d_5 = 12(28 + 25 + 22 + 24 + 27) = 1512 \text{ feet}$$